Gonality, Clifford index and multisecants

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Introduction

The gonality of a projective curve C is the minimal degree of a morphism $f : C \to \mathbf{P}^1$. It is a classical invariant which has been refined by the introduction of the Clifford index:

$$Cliff(C) := min\{Cliff(\mathcal{L}) : \mathcal{L} \in Pic(C),\ h^0(\mathcal{L}) \geq 2,\ h^1(\mathcal{L}) \geq 2\}$$

where $Cliff(\mathcal{L}) := deg(\mathcal{L}) - 2(h^0(\mathcal{L}) - 1)$ (see for example [CM] and [ELMS]).

In the first section of this paper we consider subcanonical curves in $\mathbf{P}^3$. Let $C \subset \mathbf{P}^3$ and let Γ be a point set computing the gonality of C. If $l \geq 2$ represents the maximum degree of a zero-dimensional subscheme of C which is contained in a line, then $d := Gon(C) = d(\Gamma) \leq d(C) - l$. If $Gon(C) = d(C) - l$ we will say that *Gon(C) is computable by multisecants* . In [B] Basili proved that if C is a complete intersection then $Gon(C)$ is computable by multisecants. Furthermore in the same paper, Basili computes the Clifford index of complete intersections. In this paper we generalize these results to most subcanonical curves in $\mathbf{P}^3$ (see Theorem 1.11):

Theorem. *Let E be a rank two vector bundle in $\mathbf{P}^3$. If $t >> 0$ and if C is a curve which is the zero locus of a section of $E(t)$ then $Gon(C)$ is computable by multisecants. Moreover, if C is not bielliptic, then either $Cliff(C) = Gon(C) - 3$ or $Cliff(C) = Gon(C) - 2$. Furthermore, the following conditions are equivalent:*
1) $Cliff(C) = Gon(C) - 3$;
2) $Cliff(C) = Cliff(\mathcal{O}_C(1)) = d(C) - 6$;
3) C does not have four-secant lines (i.e. $l = 3$).

Our approach is completely different from Basili's one and relies on vector bundle techniques: Bogomolov's unstability theorem [La] and Tyurin's work [T].

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In section 2 we consider a natural problem arising already from Basili's work: the stratification by multisecants of the Hilbert scheme of complete intersections. This problem is interesting by itself, not only for complete intersections, and we consider it in two extremal and opposite situations: complete intersections and rational curves. In both cases we prove that the locus of curves with a k-secant line is irreducible and of the expected dimension except when this cannot be true for trivial reasons (see Remark 2.7).

To conclude let us suggest two directions to extend the results of this paper:

investigate the stratification by multisecants for other Hilbert schemes;

determine further classes of curves with gonality computable by multisecants.

Actually this paper originated by a suggestion of Peskine that Basili's result should extend to projectively normal curves (notice that a complete intersection is both subcanonical and projectively normal).

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Gonality and Clifford index

Consider a smooth subcanonical curve $C \subset \mathbf{P}^3$ ($\omega_C \simeq \mathcal{O}_C(\alpha)$) and a point set Γ computing the gonality of C. If $l \geq 2$ represents the maximum degree of a zero-dimensional subscheme of C which is contained in a line, then $d := Gon(C) = d(\Gamma) \leq d(C) - l$.

Definition 1.1. *If Gon(C)=d(C)-l we will say that Gon(C) is computable by multisecants*

In this section we are going to prove the equality in the last formula for several classes of subcanonical curves.

We start with a technical lemma.

Lemma 1.2.
Suppose $h^1(\mathcal{I}_C(\alpha)) = 0$. *Then* $h^0(\mathcal{O}_C(\Gamma)) = h^1(\mathcal{I}_{\Gamma,\mathbf{P}^3}(\alpha)) + 1$.

Proof. Consider the sequence

$$0 \to \mathcal{I}_C(\alpha) \to \mathcal{I}_\Gamma(\alpha) \to \mathcal{I}_{\Gamma,C}(\alpha) \to 0$$

and look at the cohomology sequence. Since $h^1(\mathcal{I}_C(\alpha)) = 0$ and $H^2(\mathcal{I}_C(\alpha)) \simeq H^1(\mathcal{O}_C(\alpha)) \simeq H^0(\mathcal{O}_C)^* \simeq \mathbf{C}$ we find

$$0 \to H^1(\mathcal{I}_\Gamma(\alpha)) \to H^1(\mathcal{I}_{\Gamma,C}(\alpha)) \to \mathbf{C} \to 0$$

and the thesis follows taking account of $H^1(\mathcal{I}_{\Gamma,C}(\alpha)) \simeq H^0(\mathcal{O}_C(\Gamma))$.

Remarks 1.3. a) As in the lemma above, suppose $h^1(\mathcal{I}_C(\alpha)) = 0$. Since $h^1(\mathcal{I}_{\Gamma,\mathbf{P}^3}(\alpha)) = h^1(\mathcal{I}_{\Gamma,F}(\alpha))$ for any surface $F \subset \mathbf{P}^3$, in order that Γ contributes to the gonality of C it is necessary and sufficient that Γ fails to impose independent conditions to the linear series αH on F. From the very definition of Γ, that is from the condition $Gon(C) = d(\Gamma)$, it is clear that $h^1(\mathcal{I}_{\Gamma',F}(\alpha)) = 0$ for each proper subset $\Gamma' \subset \Gamma$. This implies that the set Γ is αH stable for any smooth surface containing it (see [T] and [P]) hence Γ is the zero locus of a section of a rank two vector bundle $\mathcal{E}$ on F sitting in a sequence like the following [T]:

$$0 \to \mathcal{O}_F \to \mathcal{E} \to \mathcal{I}_{\Gamma,F}(\alpha - f + 4) \to 0.$$

Furthermore, the fact that Γ computes the gonality of C implies that $h^0(\mathcal{O}_C(\Gamma)) = 2$ hence, by Lemma 1.1, $h^1(\mathcal{I}_{\Gamma,F}(\alpha)) = 1$ and the vector bundle $\mathcal{E}$ is uniquely determined.

b) Keep the notations of the previous remark and consider a pH stable divisor $\Gamma \subset C \subset F$ such that $h^1(\mathcal{I}_{\Gamma,F}(p)) \geq 1$. Then Tyurin's technique produces a vetor bundle $\mathcal{V}$ of rank $h^1(\mathcal{I}_{\Gamma,F}(p)) + 1$ whose general rank-two quotient $\mathcal{E}$ is a vector bundle embedded in a sequence like in the previous remark (of course now the bundle $\mathcal{E}$ is not uniquely determined):

$$0 \to \mathcal{O}_F \to \mathcal{E} \to \mathcal{I}_{\Gamma,F}(p - f + 4) \to 0.$$

Suppose now $h^1(\mathcal{I}_C(\alpha)) = 0$. The construction just outlined applies in particular to any divisor Γ computing the Clifford index of C with $p = \alpha$. Indeed it is easy to see that Γ is αH-stable: denotes by $\Gamma' \subset \Gamma$ the stable part of Γ (see [T] for the definition of stable part of a zero dimensional scheme in a smooth surface). We have $h^0(\mathcal{O}_C(\Gamma')) = h^0(\mathcal{O}_C(\Gamma))$ hence $Cliff(\Gamma') \leq Cliff(\Gamma)$ with equality iff $\Gamma' = \Gamma$.

c) Set $s(C) := min\{n : \ h^0(\mathcal{I}_C(n) \neq 0\}$ and consider a smooth surface F containing C whose degree f is bigger than $s(C)$. Suppose that there exists a smooth surface T, with $deg(T) < f$, such that $T \cap F = C \cup D$ with D integral. A theorem of Lopez (see [L]) says that the Picard group of the general surface of degree f containing C, has rank two and is generated by the hyperplane and D.

Notation. *In the sequel we will make the following assumption:*

($\circ$) *there exist smooth surfaces T,F',$f = deg(F') > deg(T)$ containing C such that $T \cap F' = C \cup D$ where D is smooth.*

By Remark c) above for the general surface F of degree f containing C we have $Pic(F) =< H, D >$.

Theorem 1.4. *Let C be a subcanonical ($\omega_C \simeq \mathcal{O}_C(\alpha)$), curve. Assume condition ($\circ$) is verified. Suppose that $\Gamma \subset C$ is a pH-stable ($p \leq \alpha$) divisor (in F) such that $d = d(\Gamma) < d(C)$ and $h^1(\mathcal{I}_{\Gamma,F}(p)) \geq 1$. Assume the following conditions are verified*

a) $h^1(\mathcal{I}_C(1)) = 0$;
b) $f < \alpha + 4$;
c) *there exists a positive integer s such that* $p - f + 4 > s + \frac{d}{sf}$;
d) $d(C) \leq 2(p - f + 2)f$.

Then Γ is a planar set and $d > (p - f + 3)f$.

Remark 1.5. Note that condition b) implies that $C^2 > 0$ in F hence $C \cdot E \geq 0$ for any effective divisor E.

Proof. By Lemma 1.2 and Remarks 1.3 we find a vector bundle $\mathcal{E}$ on F admitting a section vanishing on Γ. The discriminant of $\mathcal{E}$ is $\Delta(\mathcal{E}) = (p - f + 4)^2 f - 4d$. The hypothesis c) above implies $\Delta(\mathcal{E}) > (s\sqrt{f} - \frac{d}{s\sqrt{f}})^2$ hence, by the Bogomolov's unstability theorem (see [La] §4), there exists a divisor $Y \subset F$ ($0 \to \mathcal{O}_F(Y) \to \mathcal{E}$) such that

1) $2Y - (p - f + 4)H \in N^+(F)$ (the positive cone of F);
2) $[2Y - (p - f + 4)H]^2 > \Delta(\mathcal{E})$.

Notice that the last condition is equivalent to $d > Y \cdot [(p - f + 4)H - Y]$. If we set $X := (p - f + 4)H - Y$ we get

i) $Y - X \in N^+(F)$;
ii) $d > X \cdot Y$.

We claim that condition 1) implies that $H^0(\mathcal{O}_F(-Y)) = 0$. Indeed, $(2Y - kH) \in N^+$ implies $(2Y - kH) \cdot H > 0$ hence $Y \cdot H > 0$ and the claim follows. Hence the composite morphism

$$\mathcal{O}_F(Y) \to \mathcal{E} \to \mathcal{I}_{\Gamma,F}(p - f + 4)$$

does not vanish and $h^0(\mathcal{I}_{\Gamma,F}(p - f + 4 - Y)) \neq 0$. So we may suppose X to be an effective curve containing Γ. Moreover, combining Remark 1.5 with the fact that some multiple of $Y - X$ is effective (see [H] Corollary V 1.8), we find

$$C \cdot Y \geq C \cdot X \tag{$*$}$$

By hypothesis, $Pic(F) =< H, D >$ where $F \cap T = C \cup D$; furthermore the exact sequence of liaison:

$$0 \to \mathcal{I}_{F \cap T}(t + f - \alpha - 4) \to \mathcal{I}_D(t + f - \alpha - 4) \to \omega_C(-\alpha) \to 0$$

shows that there exists an effective divisor R on F such that $D + R = (t + f - \alpha - 4)H$. From the exact sequence above $R \cap C = \emptyset$ (because $\omega_C(-\alpha) \simeq \mathcal{O}_C$).

Furthermore $Pic(F) =< H, R >$ and $\omega_R \simeq \mathcal{O}_R(g-4)$ with $g = 2f-\alpha-4$. We have $Y \sim \beta H + \delta R$ hence $X \sim (p-f+4-\beta)H - \delta R$ and ii) says that

$$d > X \cdot Y = \beta(p-f+4-\beta)f + \delta d(R)(p-f+4-2\beta) - \delta^2 R \cdot R \qquad (**).$$

The adjunction formula provides to the selfintersection of R

$$R \cdot R = (g-f)d(R)$$

hence

$$X \cdot Y = \beta(p-f+4-\beta)f + \delta d(R)((p-f+4-2\beta) - \delta(g-f)).$$

By Lemma 1.7 stated below we find $X \cdot Y \geq \beta(p-f+4-\beta)f$ and Lemma 1.6 combined with $(**)$ implies $d(C) > d > \beta(p-f+4-\beta)f$ with $0 < \beta < p-f+4$. The last inequality, the hypothesis $d(C) \leq 2(p-f+2)f$ and the fact that the maximum of $\phi(\beta) := \beta(p-f+4-\beta)$ is reached for $\beta = \frac{p-f+4}{\beta}$ imply that either $\beta = 1$ or $p-f+4-\beta = 1$. We claim that the second item is the right one. Indeed, $Y \sim \beta H + \delta R$, $X \sim (p-f+4-\beta)H - \delta R$ and $(*)$ implies $C \cdot Y = \beta d(C) \geq C \cdot X = (p-f+4-\beta)d(C)$. Hence X is linearly equivalent to $H - \delta R$. To prove that Γ is planar it suffices to combine $\Gamma \subset X \cap C$ and $X \cdot C \sim_C H \cdot C$ with the fact that C is linearly normal. Finally, the thesis $d > (p-f+3)f$ follows directly from $(**)$ with $\beta = 1$.

Lemma 1.6. $0 < \beta < p-f+4$.

Proof.
$\beta < p-f+4$: Since $\Gamma \subset X \cap C$ we find $C \cdot X = C \cdot [(p-f+4-\beta)H - \delta R] \geq d$. But $C \cap R = \emptyset$ hence $p-f+4 > \beta$.

$0 < \beta$: $\beta d(C) = (\beta H + \delta R) \cdot C = Y \cdot C \geq X \cdot C \geq d > 0$ (as above, recall that $R \cdot C = 0$ and that $\Gamma \subset X \cap C$).

Lemma 1.7. *We have* $\delta((p-f+4-2\beta) + \delta(f-g)) \geq 0$.

Proof. We proceed in two steps depending on the sign of δ.

$\delta \geq 0$: by the last lemma $\beta < p-f+4$ hence $p-f+4-2\beta > -(p-f+4) \geq -(f-g)$ (recall that $p \leq \alpha$). The claim follows since $\delta((p-f+4-2\beta)+\delta(f-g)) \geq \delta(\delta(f-g)-(f-g)) \geq 0$ (note that hypothesis b) implies $f-g > 0$).

$\delta < 0$: by $(*)$ we have $(Y-X) \cdot C \geq 0$ hence $2\beta - p - 4 + f \geq 0$. The lemma follows.

Corollary 1.8. *Let C be a subcanonical ($\omega_C \simeq \mathcal{O}_C(\alpha)$), curve and $\Gamma \subset C$ a divisor such that $d := d(\Gamma) = Gon(C)$. Suppose condition $(\circ)$ is*

verified. Additionally, assume the conditions a)- d) of Theorem 1.4 with $\alpha = p$ and suppose $h^1(\mathcal{I}_C(\alpha)) = 0$.

Then Γ is a planar set hence $Gon(C) = d(C) - l$ i.e. $Gon(C)$ is computable by multisecants.

Proof. It follows combining Lemma 1.2, Remark 1.3 *a*) and Theorem 1.4.

Corollary 1.9. *Let C be a subcanonical ($\omega_C \simeq \mathcal{O}_C(\alpha)$), curve and $\Gamma \subset C$ a divisor. Supose condition ($\circ$) holds and assume the following conditions are verified (as above, $p \leq \alpha$)*

a) $h^1(\mathcal{I}_C(1)) = 0$;

b) $f < \alpha + 4$;

c) *there exists a positive integer s such that* $p - f + 4 > s + \frac{d(\Gamma)}{sf}$;

d) $d(C) \leq 2(p - f + 2)f$.

If $\Gamma_1 \subset \Gamma$ is a subdivisor such that $d(\Gamma_1) \leq (p - f + 3)f$ then $h^1(\mathcal{I}_{\Gamma_1,F}(p)) = 0$.

Proof. If $h^1(\mathcal{I}_{\Gamma_1,F}(p)) \neq 0$, applying Theorem 1.4 to the stable part of Γ_1 we should get $d > (p - f + 3)f$ (see [T] for the definition of stable part).

Theorem 1.10. *Let C be a subcanonical ($\omega_C \simeq \mathcal{O}_C(\alpha)$, $\alpha \geq 4$) curve and $\Gamma \subset C$ a divisor computing the Clifford index of C. Assume ($\circ$). Suppose further that C is neither hyperelliptic nor bielliptic. Set $d := d(\Gamma)$. Assume the following conditions are verified*

a) $h^1(\mathcal{I}_C(\alpha)) = h^1(\mathcal{I}_C(1)) = 0$;

b) $f < \alpha + 4$;

c) *there exists a positive integer s such that* $\alpha - f + 3 > s + \frac{d}{sf}$;

d) $d(C) \leq 2(\alpha - f + 1)f$.

Then either $Cliff(C) = Gon(C) - 3$ or $Cliff(C) = Gon(C) - 2$. Moreover, the following conditions are equivalent:

1) $Cliff(C) = Gon(C) - 3$;

2) $Cliff(C) = Cliff(\mathcal{O}_C(1)) = d(C) - 6$;

3) C does not have four-secant lines (i.e. $l = 3$).

Proof. Note that the numerical conditions are just the same of Theorem 1.4 for $p = \alpha - 1$ and that they are *a fortiori* verified for $p = \alpha$. Hence we can apply Corollary 1.8 and $Gon(C) = d(C) - l$.

Arguing as in Theoreme 4.3 of [B] it easy to show that either $Cliff(C) = Gon(C) - 2$ or $Cliff(C) = Gon(C) - 3$. Let us suppose that $Cliff(C) = Gon(C) - 3$. Then (see again Theoreme 4.3 of [B]) there exists a subdivisor $\Gamma' \subset \Gamma$ such that $Gon(C) = d(\Gamma') = d(C) - l$. We know that there exists a plane

H containing Γ'. Since $\alpha \geq 4$ we find $g(C) = \frac{\alpha d(C)+2}{2} > 2(d(C) - l - 3) + 5 = 2Cliff(C) + 5$ hence, by [CM] Cor. 3.2.5,

$$d(\Gamma) \leq \frac{3}{2}(Cliff(C) + 2). \qquad (+)$$

Let Γ_1 be the residual set to $\Gamma \cap H$ inside Γ. By (+) the degree of Γ_1 is bounded by $\frac{1}{2}d(C) - \frac{l+1}{2}$. The hypothesis $d(C) \leq 2(\alpha - f + 1)f$ implies $d(\Gamma_1) < (\alpha - f + 2)f$ hence, by Corollary 1.9 $h^1(\mathcal{I}_{\Gamma_1,F}(\alpha - 1)) = 0$.

By the following sequence

$$0 \to \mathcal{I}_{\Gamma_1,F}(\alpha - 1) \to \mathcal{I}_{\Gamma,F}(\alpha) \to \mathcal{I}_{\Gamma\cap H, H\cap F}(\alpha) \to 0$$

we have $h^0(\mathcal{O}_C(\Gamma)) = h^0(\mathcal{O}_C(\Gamma \cap H))$ (recall Lemma 1.2) hence $\Gamma \subset H$ because Γ computes $Cliff(C)$. Note that the assumption $Cliff(C) = Gon(C) - 3$ implies that $h^0(\mathcal{O}_C(\Gamma)) \geq 3$ (otherwise Γ would compute the gonality of C and $Cliff(C) = Gon(C) - 4$). On the other hand, $\Gamma \subset H$ implies $h^0(\mathcal{O}_C(\Gamma)) \leq 4$ with equality if and only if $\mathcal{O}_C(\Gamma) \simeq \mathcal{O}_C(1)$. Combining the last inequalities with $l \geq 3$ we see that the hypothesis $Cliff(\Gamma) = d(\Gamma) - 2(h^0(\mathcal{O}_C(\Gamma)) - 1) = Gon(C) - 3 = d(C) - l - 3$ implies $\mathcal{O}_C(\Gamma) \simeq \mathcal{O}_C(1)$ and $l = 3$.

To conclude the proof it suffices to show that $l = 3$ implies $Cliff(C) = Gon(C) - 3$ and $\Gamma = \mathcal{O}_C(1)$ which follows at once by $Cliff(C) \geq Gon(C) - 3 = d(C) - 6 = Cliff(\mathcal{O}_C(1))$.

Theorem 1.11. *Let $C \subset \mathbf{P}^3$ be the zero locus of a section of an high twist of a rank two vector bundle of $\mathbf{P}^3$. Then we have $Gon(C) = d(C) - l$. Suppose C is not bielliptic. Then either $Cliff(C) = Gon(C) - 3$ or $Cliff(C) = Gon(C) - 2$. Moreover, the following conditions are equivalent:*

1) $Cliff(C) = Gon(C) - 3$;

2) $Cliff(C) = Cliff(\mathcal{O}_C(1)) = d(C) - 6$;

3) C does not have four-secant lines (i.e. $l = 3$).

Proof. Let C be a curve coming as the zero locus of an high order twist of a normalized vector bundle E over $\mathbf{P}^3$:

$$0 \to \mathcal{O} \to E(t) \to \mathcal{I}_C(2t + c_1) \to 0 \quad t >> 0$$

(where either $c_1 = 0$ or $c_1 = -1$).

The assumption a) of Theorem 1.4 is satisfied:

$h^1(\mathcal{I}_C(\alpha)) = h^1(\mathcal{I}_C(2t + c_1 - 4)) = h^1(E(t - 4)) = 0$ if $t >> o$.

$h^1(\mathcal{I}_C(1)) = h^1(E(1 - t - c_1)) = 0$ if $t >> 0$.

Let r be such that $E(r)$ is globally generated, then also $\mathcal{I}_C(r + c_1 + t)$ is globally generated and condition ($\circ$) is verified for the general surface in $H^0(\mathcal{I}_C(r + c_1 + t + 1))$. Furthermore, we have $r + c_1 + t + 1 > s(C)$. Now we apply Theorem

1.10 with $f = r + c_1 + t + 1$ and we notice that if $\omega_C \sim \mathcal{O}_C(\alpha)$ with $\alpha > 0$ then C is not hyperelliptic. We verify the numerical conditions for $f = r + c_1 + t + 1$:

b): $r + t + 1 < 2t$; satisfied for $t > r$.

c): $t - r - 2 > s + \frac{d}{s(r+c_1+t+1)}$; since $d < d(C) = c_2 + c_1 t + t^2$, it is enough to check $t - r - 2 \geq s + \frac{c_2+c_1t+t^2}{s(r+c_1+t+1)}$ which is verified by $s = 2$ and $t >> 0$.

d): $c_2 + c_1 t + t^2 \leq 2(t-r-4)(r+c_1+t+1) = 2[t^2+(c_1-3)t-(r+c_1+1)(r+4)]$; it is verified for $t >> 0$.

Examples.

i) In the case of a decomposed vector bundle we can make precise computations and recover Basili's results except for complete intersections of type: $(3, a)$, $(4, 4)$ and $(5, 5)$.

ii) In the same vein if C be the zero locus of a section of $N(t)$ $(t \geq 7)$ where N is a normalized Null-Correlation bundle of $\mathbf{P}^3$ we see that the argument of the previous proof applies hence the conclusion of Theorem 1.11 holds.

Multisecants to space curves.

The results of the previous section naturally introduce the following question: for which k does there exists a smooth complete intersection curve of type (a, b) with a "maximal" k-secant line (i.e. the curve has no $l-$secant line with $l > k$)? More generally one could ask for a description of the locus of complete intersections with a $k-$secant line. This natural problem is of interest by itself, and not only for complete intersections curves. We will consider two extremal, and opposite situations: complete intersections and rational curves.

Generalities

Notations 2.1. We denote by $H(d, g)$ the closure in $Hilb(\mathbf{P}^3)$ of the set of smooth, connected curves of degree d, genus g. By $H^s(d, g)$ we will denote the open subset of $H(d, g)$ parametrizing smooth curves. Similarly if H is an irreducible component of $H(d, g)$, H^s will denote the open subset of H corresponding to smooth curves.

We will denote by Al^k the closed subscheme of $Hilb^k\mathbf{P}^3$ parametrizing zero-dimensional subschemes of length k which are contained in a line (i.e. which are "aligned"). We recall that, for $k > 1$, Al^k is smooth, irreducible, of dimension $4 + k$ (consider the natural map $Al^k \to Gr(1, 3)$).

If H is an irreducible component of $H(d, g)$ we have the incidence variety $\mathcal{I}_k(H)$ (or, if no confusion can arise $\mathcal{I}_k$): $\mathcal{I}_k := \left\{(Z, C) \in Al^k \times H \mid Z \subset C\right\}$; associated to $\mathcal{I}_k$ we have the diagram:

$$\begin{array}{ccc} \mathcal{I}_k & \xrightarrow{p_k} & H \\ \downarrow q_k & & \\ Al^k & & \end{array}$$

where p_k, q_k denote the natural projections.

In this situation we define $\mathcal{I}_k^s$ as $p_k^{-1}(H^s)$. Clearly $\mathcal{I}_k^s$ is open in $\mathcal{I}_k$.

Definition 2.2. *The locus of curves of H with a $k-$secant line is $H_k := p_k(\mathcal{I}_k)$. The locus of smooth curves of H with a $k-$secant line is $H_k^s := H_k \cap H^s = p_k(\mathcal{I}_k^s)$.*

Remarks 2.3.

(i) A line L is a k-secant to the smooth connected curve C (of degree > 1) if *length*$(C \cap L) \geq k; L$ is a *proper* $k-$*secant* if *length*$(C \cap L) = k, L$ is a *maximal secant* to C if for $l > k, C$ has no $l-$secant lines.

(ii) It may happen that $\mathcal{I}_k$ is empty. It may also happen that $\mathcal{I}_k \neq \emptyset$ while $\mathcal{I}_k^s$ is empty.

(iii) To pass through one point impose two conditions to curves in H, so we may expect, in general, the general fiber of q_k to be of dimension $h - 2k$ ($h = \dim(H)$), hence we may expect $\dim(\mathcal{I}_k) = h - (k - 4)$. Also, in general, and if $k > 3$, we may expect the general fiber of p_k to be finite, and thus $\dim(H_k) = h - (k - 4)$. Of course we are mainly interested in smooth curves. We will say that (an irreducible component of) H_k or H_k^s has the *expected dimension* if it is of dimension $h - (k - 4)$.

The following general statement will be quite useful:

Proposition 2.4. *With notations as above, let H'_k, H'_{k+1} be irreducible components of H_k, H_{k+1} such that $H'_{k+1} \subset H'_k$. Set $\mathcal{I}'_k = p_k^{-1}(H'_k)$. Assume* $\dim(\mathcal{I}'_k) = h - (k - 4)$ *and* $\dim(H'_{k+1}) = h - (k - 3)$ *(i.e. $\mathcal{I}'_k$ and H'_{k+1} are both of the expected dimension). Furthermore assume $\mathcal{I}'_k$ smooth. Then H'_k also has the expected dimension:* $\dim(H'_k) = h - (k - 4)$.

Proof. Since $H'_{k+1} \subset H'_k$, $\dim(H'_k) \geq \dim(H'_{k+1}) = h - (k - 3)$. Since $\dim(\mathcal{I}'_k) = h - (k - 4)$, if $H'_k = p_k(\mathcal{I}'_k)$ has not the expected dimension then $\dim(H'_k) = \dim(H'_{k+1})$, and since they are both irreducible, $H'_k = H'_{k+1}$. Moreover, for general C in $H'_k, p_k^{-1}(C)$ has dimension one. By generic smoothness we may assume that the general fiber $p_k^{-1}(C)$ is a smooth, equidimensional curve. On the other hand, $C \in H'_{k+1}$, this means that among the k-secants to C there is at least one $(k + 1)$-secant. This $(k + 1)$-secant will count several time as a k-secant. in other words $p_k^{-1}(C)$ is singular at points (Z, C) where Z is on a $(k + 1)$-secant line to C; this contradicts the smoothness of $p_k^{-1}(C)$ for general C

Complete intersections

Abusing notations we will denote by $H(a,b)$ the Hilbert scheme of complete intersections of type $(a,b), a \leq b$. As it is well known $H(a,b)$ is integral of dimension $h(a,b)$ where $h(a,b) = h^0(\mathcal{O}_{\mathbf{P}}(a)) + h^0(\mathcal{O}_{\mathbf{P}}(b)) - h^0(\mathcal{O}_{\mathbf{P}}(b-a)) - 2$, if $a < b$, and where $h(a,b) = 2.h^0(\mathcal{O}_{\mathbf{P}}(a)) - 4$ if $a = b$.

Lemma 2.5. *If $k \leq b$ the morphism $q_k : \mathcal{I}_k(a,b) \to Al^k$ is surjective, smooth, of relative dimension $max\{h(a,b)-2k, h(a,b)-(a+1)-k\}$. In particular if $k \leq a+1$ $\mathcal{I}_k(a,b)$ is smooth, irreducible, of dimension $h(a,b)-(k-4)$.*

Proof. This follows essentially from the fact that every Z in Al^k is a complete intersection $(1,1,k)$ and hence gives independant conditions to forms of degree $\geq k-1$.

Lemma 2.6. *If $k = b$ and $a \geq 4$, $\mathcal{I}_b^s$ is non-empty and $p_b : \mathcal{I}_b^s \to H_b^s(a,b)$ is generically finite (in fact birational if $a < b$). In particular $H_b^s(a,b)$ is irreducible of dimension $h(a,b)-(a-3)$.*

Proof. First assume $a < b$. Take a line $L \subset \mathbf{P}^3$. By [L] (see also Remark 1.3.c), if F_a is a sufficiently general surface of degree $a \geq 4$, containing L then $Pic(F_a)$ is generated by L and the hyperplane section, in particular F_a doesn't contain any further line. Indeed if R is another line on F_a then $R \sim cH + dL$. From $H \cdot R = 1$ we get $d = 1 - ca$. Since $L^2 = 2 - a$, $R \cdot L = c + (ac-1)(a-2)$, but since $0 \leq R \cdot L \leq 1$, we get a contradiction. We may assume F_a smooth. Let $C = F_a \cap F_b$, F_b a general surface of degree b. Then C is smooth and L is a (proper) $b-$secant to C. Now C has no other k-secants, $k > a$; indeed such a secant would have to lie on F_a.

The case $a = b$ requires an extra argument.

Let C be a smooth complete intersection of type (b,b) and let R be a b-secant to C. Observe that R is a proper b-secant to C. Moreover from the exact sequence

$$0 \to \mathcal{I}_{C \cup R} \to \mathcal{I}_C \to \mathcal{O}_R(-b) \to 0$$

it follows that $h^0(\mathcal{I}_{C\cup R}(b)) = 1$. So, to any b-secant, R, there is associated one surface, F_R, of the pencil $\mathbf{P}(H^0(\mathcal{I}_C(b)) \simeq \mathbf{P}^1$. If C has infinite b-secants we get a morphism from (a component of) the curve, Λ, of $b-$secants to $\mathbf{P}^1 : \varphi : \Lambda \to \mathbf{P}^1 : R \to F_R$. If φ is constant then C lies on a degree b surface of $b-$secants lines, otherwise φ is surjective and *every* $F \in H^0(\mathcal{I}_C(b))$ contains a b-secant line to C. We will show that this cannot happen if C is sufficiently general.

Indeed take a general surface, F_b, of degree $b \geq 4$. We may assume that F_b doesn't contain any line (Noether-Lefschetz theorem). Now let $F_b^{'}$ be a general surface of degree b containing the line L. We may assume $X = F_b \cap F_b^{'}$ smooth.

Clearly L is a b-secant to X. Assume that every fiber of $p_b : \mathcal{I}_b^s(b,b) \to H_b^s(b,b)$ has dimension one. By Lemma 2.5 and by generic smoothness we may assume that the general fiber of p_b is a smooth, equidimensional curve. Since curves as X above are clearly general in $H_b^s(b,b)$, we may assume, with notations as above, that $L \in \Lambda$ (i.e. L is not an isolated point of $p_b^{-1}(X)$). Then we get a contradiction. Indeed, on one hand the morphism φ has to be constant because F_b doesn't contain any line. On the other hand if $F = \lambda F_b + \mu F_b'$ is the surface of b-secants corresponding to the point $\varphi(\Lambda)$, then $L \subset F \cap F_b' = X$, which is absurd (notice that $F \neq F_b'$ because by [L] we may assume that L is the only line on F_b')

Remark 2.7. Observe that $H_k^s(a,b) = H_{a+1}^s(a,b)$ if $k \geq a+1$, indeed if $C = F_a \cap F_b$ then every k-secant to C is contained in F_a and thus is b-secant to C.

Corollary 2.8. *For $a \geq 4$ and $4 \leq k \leq b, H_k^s(a,b)$ is (non empty) integral, of the expected dimension $h(a,b) - (k-4)$ if $4 \leq k \leq a+1$ and of dimension $h(a,b) - (a-3)$ if $a+1 \leq k \leq b$.*

Proof. Since every b-secant is a k-secant for $k \leq b$, by Lemma 2.6 we get: $\mathcal{I}_k^s(a,b) \neq \emptyset$. Clearly $\mathcal{I}_k^s(a,b)$ is open in $\mathcal{I}_k(a,b)$. By Lemma 2.5 we conclude that $H_k^s(a,b)$ is irreducible. It remains to compute the dimension of $H_k^s(a,b)$. For $a+1 \leq k \leq b$ we combine Lemma 2.6 and Remark 2.7. If $4 \leq k \leq a+1$ we argue by descending induction on k using Proposition 2.4, the starting point being the case $k = a+1$ (combining Lemma 2.6 and Remark 2.7). The assumptions of Prop. 2.4 are satisfied since $H_{k+1}^s(a,b) \subset H_k^s(a,b)$ and both are irreducible, and since $\mathcal{I}_k^s(a,b)$ is smooth (Lemma 2.5).

Remark 2.9. If $a < 4$, our arguments break down, but a detailed analysis of curves on low degree surfaces should give a precise description of $H_k^s(a,b)$.

Rational curves

The starting point is the following basic remark:

Remark 2.10. Let δ be a ∞^3 linear system of degree d divisors on $\mathbf{P}^1$. Assume δ very ample so that δ yields an immersion $f : \mathbf{P}^1 \to \mathbf{P}^3$. The points $f(x_1), ..., f(x_k)$ of $\mathbf{P}^3$ will be aligned if and only if δ contains a pencil having $x_1, ..., x_k$ in its base locus.

Effective divisors of degree d on $\mathbf{P}^1$ are parametrized by $\mathbf{P} := \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(d)) \simeq \mathbf{P}^d$, so a pencil is a line in $\mathbf{P}$, i.e. a point in $G(1, \mathbf{P})$. We will denote by $\mathcal{P}_k \subset G(1, \mathbf{P})$, the locus of pencils with base locus of length $\geq k$.

Lemma 2.11. *With notations as above, $\mathcal{P}_k$ is irreducible, of dimension $2d-k-2$.*

Proof. Fix an effective divisor of degree k on $\mathbf{P}^1 : D = x_1 + ... + x_k$. The effective divisors of degree d containing D build a $\mathbf{P}^{d-k}$ in $\mathbf{P}$, let's denote it by $\mathbf{P}^{d-k}(D)$. A line in $\mathbf{P}^{d-k}(D)$ is a pencil having $x_1, ..., x_k$ in its base locus. Now $\mathcal{P}_k$ can be described as follows: it is the image in $G(1,\mathbf{P})$ of a fibration over $Hilb^k\mathbf{P}^1 \simeq \mathbf{P}^k$ with fibers isomorphic to $G(1, d-k)$. Let's try to be more precise. Consider the natural exact sequence:

$$0 \to K \to H^0(\mathcal{O}_{\mathbf{P}^1}(k)) \otimes H^0(\mathcal{O}_{\mathbf{P}^1}(d-k)) \to H^0(\mathcal{O}_{\mathbf{P}^1}(d)) \to 0$$

We have the Segre embedding: $\mathbf{P}^k \times \mathbf{P}^{d-k} \hookrightarrow S \subset \mathbf{P}^N, N = k(d-k)+d$, where $\mathbf{P}^N \simeq \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^1}(k)) \otimes H^0(\mathcal{O}_{\mathbf{P}^1}(d-k)))$, the projection of S from $\mathbf{P}(K)$ yields a finite morphism $S \to \mathbf{P}$ which presents $\mathbf{P}$ as ruled by the $\mathbf{P}^{d-k}$-fibers of S. This defines a diagram:

$$\begin{array}{ccc} S & & \\ \downarrow & & \\ \mathbf{P}^k & \to & Z \subset G(d-k,\mathbf{P}) \end{array}$$

where Z is irreducible of dimension k. Now in $G(1,\mathbf{P}) \times G(d-k,\mathbf{P})$ consider the incidence variety: $P := \{(L,E) \in G(1,\mathbf{P}) \times Z / L \subset E\}$, clearly P is irreducible, of dimension $k + \dim G(1, d-k)$. Finally $\mathcal{P}_k = \phi(P)$, where $\phi : P \to G(1,\mathbf{P})$ is the projection. To conclude we observe that ϕ is birational: let $L \subset \mathbf{P}^{d-k}(D), L$ generated by the two points $D + D_{d-k}, D + D'_{d-k}$, where D is a given effective divisor of degree k; if $Supp(D) \cap Supp(D_{d-k}) = Supp(D) \cap Supp(D'_{d-k}) = Supp(D_{d-k}) \cap Supp(D'_{d-k}) = \emptyset$, then L cannot be contained in $\mathbf{P}^{d-k}(D')$ for $D \neq D'$

We recall the following classical fact:

Lemma 2.12. *For $d \geq 3$ there exists a smooth rational curve $C \subset \mathbf{P}^3$ with a $(d-1)-$secant line. Moreover if C has more than one $(d-1)-$ secant lines, then C lies on a smooth quadric surface. Hence for $d \geq 5$, there exists a smooth rational curve of degree d in $\mathbf{P}^3$ with exactly one $(d-1)-$secant line.*

Proof. We may assume $d \geq 5$. For the existence look at rational curves on the cubic scroll $S \subset \mathbf{P}^4$. Then project from a general point. The image $\overline{S}$ of S is a cubic surface with a double line, the double line will be the $(d-1)-$secant line. Finally observe that if $C \subset \overline{S}$ then $h^0(\mathcal{I}_C(2)) = 0$. Indeed, if $d > 6$ this is just by degree reasons, if $5 \leq d \leq 6$ and if $h^0(\mathcal{I}_C(2)) \neq 0$, then C would have

to be projectively normal (because it would have to be a complete intersection if $d=6$, or linked to a line if $d=5$), and this is absurd.

Finally if R, L are two $(d-1)-$secant lines to C, then $R \cap L = \emptyset$ (otherwise the plane $< R, L >$ will intersect C in too many points). Let D be a three secant line to C, D is disjoint from R and L. Let Q be the quadric generated by R, L and D, then Q intersects C in $2d+1$ points, hence $C \subset Q$.

Lemma 2.13. *For* $4 \leq k \leq d-1$, $H^s_k(d,0)$, *the locus of smooth rational curves of degree d with a* $k-$*secant line, is irreducible. Moreover* $H^s_{d-1}(d,0)$ *has the expected dimension* $3d+5$.

Proof. First of all notice that, by Lemma 2.12, $H^s_k(d,0) \neq \emptyset$. A smooth rational curve of degree d in $\mathbf{P}^3$ with a $k-$ secant line corresponds to a ∞^3-linear system containing a pencil having k points in its base locus (see Remark 2.10). In $G(1,\mathbf{P}) \times G(3,\mathbf{P})$, consider the (restricted) incidence variety: $I \subset \mathcal{P}_k \times G(3,\mathbf{P})$, $I = \{(L,E)/L \subset E\}$ and the associated diagram: $\mathcal{P}_k \xleftarrow{q} I \xrightarrow{p} G(3,\mathbf{P})$. Since the fibers of q are isomorphic to $G(1, d-2)$, by Lemma 2.11, we conclude that I is irreducible, of dimension $4d-k-8$. It follows that $p(I)$ is irreducible. As noticed at the beginning of the proof, the intersection, in $G(3,\mathbf{P})$, of $p(I)$ with the open set of very ample ∞^3 linear systems is non empty. Let U denote this intersection. Choosing a basis in the $4-$vector space corresponding to an ∞^3-linear system, yields a Stiefel fibration $\mathcal{S} \rightarrow U$, with fibers isomorphic to $Aut(\mathbf{P}^3)$, now $Aut(\mathbf{P}^1)$ acts (fiberwise) on $\mathcal{S}$ and the quotient is $H^s_k(d,0)$. It follows that $H^s_k(d,0)$ is irreducible. From this description it also follows that $\dim(H^s_k(d,0)) = \dim(p(I)) + 12$, in particular $H^s_k(d,0)$ has the expected dimension if and only if p is generically finite, which amounts to say that the generic rational curve of degree d with a $k-$secant line has only a finite number of $k-$secant lines. If $k=d-1$, from Lemma 2.12, p is birational and we are done (for $k < d-1$ we didn't find a simple direct argument).

In the next lemma notations are as in 2.1.

Lemma 2.14. *For* $d \geq 5$ *and* $4 \leq k \leq d-1$, $\mathcal{I}^s_k$ *is smooth, of the expected dimension* $4d-(k-4)$.

Proof. Consider the incidence variety

$$\mathbf{I}_k = \big\{(Z,C) \in Hilb^k\mathbf{P}^3 \times H^s(d,0)/Z \subset C\big\}$$

and the corresponding diagram:

$Hilb^k\mathbf{P}^3 \xleftarrow{f} \mathbf{I}_k \xrightarrow{\pi} H^s(d,0)$. Since $H^s(d,0)$ is smooth at $[C]$, if the natural map $r: H^0(N_C) \rightarrow H^0(Z, N_C \mid Z)$ is surjective then f is smooth at (Z,C) (see [K] and also [Pe]). Since $N_C \simeq \mathcal{O}_{\mathbf{P}^1}(a) \oplus \mathcal{O}_{\mathbf{P}^1}(b)$, with $d+2 \leq a$,

$d+2 \leq b$, we have $h^1(N_C(-Z))=0$ and r is surjective. We conclude that f is smooth. Since $\mathbf{I}_k$ is irreducible of dimension $4d+k$, and since f is dominant [Pe], we conclude that f has relative dimension $4d-2k$. Now $\mathcal{I}_k^s$ is obtained by base-change:

$$\begin{array}{ccc} \mathcal{I}_k^s & \rightarrow & \mathbf{I}_k \\ q \downarrow & & \downarrow f \\ Al^k & \hookrightarrow & Hilb^k\mathbf{P}^3 \end{array}$$

Since $\mathcal{I}_k^s \neq \emptyset$ (Lemma 2.12), it follows that q is smooth, of relative dimension $4d-2k$. Observe that q is surjective (take a smooth quadric, Q, containing the line $< Z >$ and look at the linear system of rational curves of degree d on Q passing through Z). So, since Al^k is smooth, of dimension $4+k$, $\mathcal{I}_k^s$ is smooth, of dimension $4d-(k-4)$.

Theorem 2.15. *For $d \geq 5$ and $4 \leq k \leq d-1$, $H_k^s(d,0)$ is integral, of the expected dimension $4d-(k-4)$. Moreover if C corresponds to a general point of $H_k^s(d,0)$ then C has a finite number, $s(k)$, of $k-$secant lines (which are all proper and maximal).*

Proof. The proof is by descending induction on k. The case $k=d-1$ follows from Lemma 2.12 and Lemma 2.13. Assume the theorem for $k+1$ and suppose $\dim(H_k^s(d,0)) < 4d-(k-4)$. Since $H_{k+1}^s(d,0) \subset H_k^s(d,0)$ and since they are both irreducible, it follows that $H_k^s(d,0) = H_{k+1}^s(d,0)$, furthermore, for any irreducible component, $\overline{\mathcal{I}}_k^s$ of $\mathcal{I}_k^s$, $\overline{\mathcal{I}}_k^s \rightarrow H_k^s(d,0)$ has one-dimensional fibers. Since $\mathcal{I}_k^s$ is smooth (Lemma 2.14), we can repeat the argument of Prop. 2.4 and get a contradiction. This proves that $H_k^s(d,0)$ has the expected dimension. Hence $H_{k+1}^s(d,0)$ is a proper closed subset of $H_k^s(d,0)$. So if C corresponds to a general point of $H_k^s(d,0), C$ has no $(k+1)-$secant lines. Moreover for every irreducible component $\overline{\mathcal{I}}_k^s$ of $\mathcal{I}_k^s$, $\overline{\mathcal{I}}_k^s \rightarrow H_k^s(d,0)$ is generically finite. Since there are a finite number of irreducible components, C has only finitely many $k-$secant lines.

Remarks 2.16. (i) We recover the classicaly known fact that every rational quintic curve has a $4-$secant line.

(ii) We have $s(d-1)=1$ (note the analogy with the complete intersections case), while $s(4)$ may be recover from the formula for quadrisecant lines. As far as we know it is still an open problem to determine $s(k)$ in general.

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